"On the Calculation of the Trajectories of Shot." By W. D. Niven, M.A., F.R.A.S. Communicated by J. Clerk Maxwell, F.R.S., Professor of Experimental Physics in the University of Cambridge. Received March 24, 1876*.

In the present state of our knowledge of the resistance of the air to shot, the problem of integrating the equations of motion of the shot and of plotting-out a representation of the curve described by it is peculiar, because, according to the best experiments we possess, the law of the retardation cannot be expressed by a single exact formula which is available for the solution. We are therefore compelled to give a solution adapted to Tables, the magnitudes of the retardation being set down in those Tables for velocities which are common in practice. The formulæ given by Hutton and by Didion, even if they were true, apply only to spherical shot; and though they are very simple formulæ, the solutions obtained by means of them are not satisfactory—first, by reason of their complexity, and next on account of the rough approximations which characterize the proofs.

Prof. Hélie, who gives an account of Didion's method in his 'Traité de Balistique,' says that it gives results which are not in accordance with fact. The fault may probably be laid in a great measure to the charge of the formula; for there can be no doubt that Mr. Bashforth's method of experimenting with his chronograph and screens gives more trustworthy and more extensive information than the ballistic pendulum experiments of Hutton and Didion; and Hutton's formula, as well as Didion's, agrees with Mr. Bashforth's Tables only for a limited range of velocities.

Mr. Bashforth himself makes no attempt to condense his Tables into concise formulæ. Accordingly, as we shall presently see, he adopts a solution which is capable of being employed in conjunction with his Tables. He divides the trajectory into small arcs, and finds for every arc the time and the horizontal and vertical distances from one end of the arc to the other. The entire trajectory may then be plotted out, and the whole time and range may be discovered, as well as the final inclination of the direction of motion. The amount of labour, however, in calculating all the quantities for a single component arc, even with the aid of copious tables, is so great that I was led to examine whether any thing could be done towards simplifying the solution and reducing the amount of calculation. It will appear in the sequel that rules of comparatively easy application can be employed, and that the tables necessary for their use are already existing or can be easily formed.

Meanwhile, without entering into details, it will be convenient to give a brief account of the drift and scope of what is attempted in this paper.

* Read May 11, 1876. See vol. xxv. p. 18.

The work is arranged under three heads, which are called the First, the Second, and the Third Methods, intended to signify three distinct and different solutions of the problem of finding the motion of shot.

The First, which is the one adopted in Mr. Bashforth's treatise, is a solution on the assumption that the retardation due to air-resistance is μv^3, where μ is a constant. Now in the actual case μ is not constant; and therefore, in dealing with the equations of motion over any component part of the trajectory, a mean value of μ must be used.

In contrast to the first method, the third adopts the mean value of another quantity, viz. the inclination of the direction of motion. The same thing was done in General Didion's solution, in which it was necessary to employ the mean value of the cosine of the inclination. It will be found that in selecting this mean we really implicate more quantities than μ. But then there will be this advantage, that whereas there is no way of determining the mean of μ by the first method, and the greatest uncertainty prevails regarding it when comparatively large arcs are integrated over, according to the work now presented the nature of the mean is investigated. In fact the determination of the mean angle may be said to be the chief object and point in these investigations, because it will be seen that there is no difficulty in establishing any of the equations which will be used, and, excepting the labour, no difficulty in forming the Tables.

What is described as the Second Method, although a distinct solution when the retardation is formulated by μv^n, is to be regarded as a mere stepping-stone to the one which follows it. It is in the second method that all the quantities are expressed in terms of the mean inclination and the magnitude of that angle determined. The quantity μ is taken constant; and therefore, in the case when $n=3$, if we use the same mean value of μ as in the first method, we ought to get a solution much the same. Beyond this, the second method possesses no further practical value in the business now in hand, being entirely superseded in that respect. Its chief value and importance consist in the determination of the mean angle, because it is shown that the same value of that mean may be used in the third method.

This point being settled, it will be found that, according to the latter method, all the required quantities will depend on three integrals, two of which (the space- and time-integrals) have already been tabulated by Mr. Bashforth. The third, which may be called the velocity-integral, is now also tabulated for ogival-headed shot, and will be found further on.

§ 2. It seems convenient at the outset to define the symbols which will be employed throughont the work:—

v will denote the velocity of the shot at any point of the trajectory;

u, the horizontal component of v;

ϕ, the inclination of the direction of motion to the horizontal line in the plane of the trajectory (the deflection from the plane being neglected);

t, the time;
x, the horizontal distance from some fixed point;
y, the vertical distance.

The integrations will be performed over a component arc of the trajectory, and the three last quantities are measured from the beginning of the arc. The values of t, x, and y over the whole arc will be denoted by T, X, and Y. The values of u at the beginning and end of the arc will be denoted by p and q; those of ϕ by α and β. The acceleration due to gravity is denoted by g.

First Method.

§ 3. The solution adopted by Mr. Bashforth is the famous one first given by John Bernoulli and published in 1721. It applies to any retardation formulated by μv^n. All the characteristic quantities are expressed in terms of ϕ, which may be accomplished briefly thus:—

The equations of motion in the horizontal direction and in the direction of the normal to the trajectory in the ascending branch are

$$\frac{du}{dt}=-\mu v^n \cos\phi, \quad \ldots \quad (1)$$

$$v\frac{d\phi}{dt}=-g\cos\phi. \quad \ldots \quad (2)$$

Hence

$$\frac{du}{d\phi}=\frac{\mu}{g}v^{n+1} \quad \ldots \quad (3)$$

$$=\frac{\mu}{g}u^{n+1}\sec^{n+1}\phi.$$

Integrate this, recollecting that the initial values of u and ϕ are p and α, and get

$$\frac{1}{u^n}-\frac{1}{p^n}=\frac{\mu}{g}\int_\phi^\alpha n\sec^{n+1}\phi\, d\phi.$$

If the symbol P_ϕ denotes $\int_0^\phi n\sec^{n+1}\phi\, d\phi$, the equation will become

$$\frac{1}{u^n}-\frac{1}{p^n}=\frac{\mu}{g}(P_\alpha-P_\phi).$$

The horizontal velocity at the end of the arc is therefore given by

$$\frac{1}{q^n}=\frac{1}{p^n}+\frac{\mu}{g}(P_\alpha-P_\beta). \quad \ldots \quad (A)$$

At the vertex of the trajectory the velocity u_0 is given by

$$\frac{1}{u_0^n}=\frac{1}{p^n}+\frac{\mu}{g}\mathrm{P}_\alpha;$$

$$\therefore\quad \frac{1}{u^n}=\frac{1}{u_0^n}\left\{1-\frac{\mu u_0^n}{g}\mathrm{P}_\phi\right\}.$$

If we put

$$\gamma=\frac{\mu u_0^n}{g}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we have

$$u=\frac{u_0}{\left(1-\gamma \mathrm{P}_\phi\right)^{\frac{1}{n}}}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From equation (2),

$$\mathrm{T}=\int_\beta^\alpha \frac{v}{g}\sec\phi\, d\phi$$

$$=\int_\beta^\alpha \frac{u}{g}\sec^2\phi\, d\phi;$$

by (5),

$$=\frac{u_0}{g}\int_\beta^\alpha \frac{\sec^2\phi\, d\phi}{\left(1-\gamma \mathrm{P}_\phi\right)^{\frac{1}{n}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{B})$$

Again,

$$\frac{dx}{dt}=u;$$

$$\therefore\quad x=\int u\,dt=-\int \frac{u^2}{g}\sec^2\phi\, d\phi;$$

$$\therefore\quad \mathrm{X}=\frac{u_0^2}{g}\int_\beta^\alpha \frac{\sec^2\phi}{\left(1-\gamma \mathrm{P}_\phi\right)^{\frac{2}{n}}}\,d\phi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{C})$$

Similarly,

$$\mathrm{Y}=\frac{u_0^2}{g}\int_\beta^\alpha \frac{\tan\phi\sec^2\phi\, d\phi}{\left(1-\gamma \mathrm{P}_\phi\right)^{\frac{2}{n}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{D})$$

§ 4. The law of retardation found by Mr. Bashforth, in his own symbols, is expressed by $2bv^3$, where

$$2b=\frac{d^2}{\mathrm{W}}\frac{\mathrm{K}}{(1000)^3},$$

d being the diameter of the cross section of the shot in inches, W its weight in pounds, and K a quantity which is a function of the velocity, and whose values are tabulated between certain values of the velocity for every ten feet. In integrating the equations of motion over any arc, the

quantity K, and therefore $2b$, is taken constant, its mean value over the arc as near as it can be guessed at being used.

Taking this law, the above results, written in the order in which they would be used by a calculator, are as follows :—

$$\left(\frac{1000}{q}\right)^3-\left(\frac{1000}{p}\right)^3=\frac{d^2}{W}\frac{K}{g}(P_\alpha-P_\beta), \quad . \quad . \quad . \quad . \quad \text{(a)}$$

$$\left(\frac{1000}{u_0}\right)^3-\left(\frac{1000}{p}\right)^3=\frac{d^2}{W}\frac{K}{g}P_\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(b)}$$

$$\gamma=\frac{d^2}{W}\frac{K}{g}\left(\frac{u_0}{1000}\right)^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(c)}$$

$$T=\frac{u_0}{g}\int_\beta^\alpha\frac{\sec^2\phi\,d\phi}{(1-\gamma P_\phi)^{\frac{1}{3}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(d)}$$

$$X=\frac{{}^2_0}{g}\int_\beta^\alpha\frac{\sec^2\phi\,d\phi}{(1-\gamma P_\phi)^{\frac{2}{3}}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(e)}$$

$$Y=\frac{u_0^2}{g}\int_\beta^\alpha\frac{\tan\phi\sec^2\phi\,d\phi}{(1-\gamma P_\phi)^{\frac{2}{3}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(f)}$$

The last three integrals Mr. Bashforth has given ample tables for, corresponding to different values of γ and between certain ranges of angle. If there is no table for the exact value of γ which results from equation (c), a method of interpolation must be employed. The integral P_ϕ is in this case $3\tan\phi+\tan^3\phi$, and its values are tabulated as well as those of its logarithm.

It will thus be seen that there are six distinct operations of some length, the first being the most serious, because there is some difficulty in getting K right. Supposing, however, that point to be settled—and I shall afterwards offer a few observations which will, I think, make the solution of (a) easier—the quantity γ must be found. It will be seen that (b) and (c) are mere stepping-stones to the time- and distance-integrals.

I shall now enter into an examination of the equations of motion, with the object of proving other formulæ, which, when we have once discovered the velocity at the end of the arc, will give the time and distances with two operations.

Second Method.

§ 5. I remark, first, that although Bernoulli's solution succeeds in expressing all the quantities in terms of integrals of ϕ, yet, owing to the difficulty and complexity of the integrals, it is practically valueless, except in two cases—first when $n=1$, and next when $n=3$. I doubt, moreover, whether, in the case $n=3$, it is the best solution when μ is not constant,

and its mean value must therefore be taken over a small arc. The tables being such as they are, there can be no doubt that, if it were possible, the most convenient independent variable would be the velocity itself. On the other hand, all attempts at humouring the equations of motion so as to introduce the velocity as independent variable are of no avail. When the trajectory is very flat, it may be possible to get results which are not very objectionable; but no general solution is hereby attainable. The forms of the equations, however, suggest as a possibly good substitute the horizontal component of the velocity. Accordingly, taking this quantity as independent variable, I now proceed to find the distance-integrals by a method of approximation.

Since

$$\frac{dx}{dt}=u,$$

and

$$\frac{du}{dt}=-\mu v^n\cos\phi,$$

$$\therefore\quad \frac{dx}{du}=-\frac{1}{\mu}\cos^{n-1}\phi\,\frac{1}{u^{n-1}};$$

$$\therefore\quad \mathrm{X}=\frac{1}{\mu}\int_q^p\cos^{n-1}\phi\,\frac{du}{u^{n-1}}.$$

Similarly,

$$\mathrm{Y}=\frac{1}{u}\int_q^p\sin\phi\cos^{n-2}\phi\,\frac{du}{u^{n-1}}.$$

§ 6. Our business now is to substitute for ϕ its value in terms of u. To enable us to do this, put $\phi=\alpha-\psi$ and expand in powers of ψ: we have

$$\mathrm{P}_\alpha-\mathrm{P}_\phi=\frac{d\mathrm{P}}{d\alpha}\psi-\text{ etc.};$$

$$\therefore\quad \frac{1}{u^n}-\frac{1}{p^n}=\frac{\mu}{g}\left(n\sec^{n+1}\alpha\psi-\frac{n(n+1)}{2}\sec^{n+1}\alpha\tan\alpha\psi^2+\text{ etc.}\right). \tag{6}$$

Also

$$\left.\begin{aligned}\cos^{n-1}\phi=\cos^{n-1}\alpha&+(n-1)\cos^{n-2}\alpha\sin\alpha\psi\\&+\frac{n-1}{2}\{(n-2)\cos^{n-3}\alpha\sin^2\alpha-\cos^{n-1}\alpha\}\psi^2\ .\\&+\text{etc.}\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\end{aligned}\right\}(7)$$

And

$$\left.\begin{aligned}\sin\phi\cos^{n-2}\phi=&\sin\alpha\cos^{n-2}\alpha+\{(n-2)\sin^2\alpha\cos^{n-3}\alpha-\cos^{n-1}\alpha\}\psi\\&+\{(n-2)(n-3)\sin^2\alpha-(3n-5)\cos^2\alpha\}\sin\alpha\cos^{n-4}\alpha\,\frac{\psi^2}{2}\\&+\text{etc.}\ \ldots\ldots\end{aligned}\right\}. \quad (8)$$

I now propose to neglect the squares and higher powers of ψ. The effect of this on the (c) integral will be that we have now to find X from

$$\frac{\cos^{n-1}\alpha}{\mu}\int_q^p\frac{du}{u^{n-1}}+\frac{(n-1)\cos^{n-2}\alpha\sin\alpha}{\mu}\int_q^p\psi\frac{du}{u^{n-1}}.$$

Call the two integrals in this expression Q and R. Then

$$Q=\frac{1}{n-2}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right);$$

and, taking account of equation (6),

$$\begin{aligned}R&=\frac{g}{\mu}\frac{\cos^{n+1}\alpha}{n}\int_q^p\left(\frac{1}{u^{2n-1}}-\frac{1}{p^n}\frac{1}{u^{n-1}}\right)du\\&=\frac{g}{\mu}\frac{\cos^{n+1}\alpha}{n}\left\{\frac{1}{2n-2}\left(\frac{1}{q^{2n-2}}-\frac{1}{p^{2n-2}}\right)-\frac{1}{n-2}\frac{1}{p^n}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right)\right\}.\end{aligned}$$

Let this be put equal to

$$Q(\alpha-\beta)f.$$

Then, since

$$\alpha-\beta=\frac{g}{\mu}\frac{\cos^{n+1}\alpha}{n}\left(\frac{1}{q^n}-\frac{1}{p^n}\right),$$

we see that

$$f=\frac{\frac{1}{2n-2}\left(\frac{1}{q^{2n-2}}-\frac{1}{p^{2n-2}}\right)-\frac{1}{n-2}\frac{1}{p^n}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right)}{\frac{1}{n-2}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right)\left(\frac{1}{q^n}-\frac{1}{p^n}\right)}.$$

Now let $\frac{1}{q}-\frac{1}{p}=\lambda$, so that $p\lambda\left(=\frac{p}{q}-1\right)$ may be regarded as a fraction whose square may be neglected. We get

$$f=\frac{\frac{n}{2}(p\lambda)^2+\frac{3n^2-7n}{6}(p\lambda)^3+\frac{7n^3-36n^2+47n}{24}(p\lambda)^4+\ldots}{n(p\lambda)^2\left\{1+(n-2)p\lambda+\frac{7n^2-32n+37}{12}(p\lambda)^2+\ldots\right\}}, \quad . \quad . \quad (9)$$

$$=\tfrac{1}{2}\frac{1+\frac{3n-7}{3}p\lambda}{1+(n-2)p\lambda}\ \text{if we neglect}\ (p\lambda)^2;$$

$$\therefore\quad f=\tfrac{1}{6}\frac{3q+(3n-7)(p-q)}{q+(n-2)(p-q)}$$

$$=\tfrac{1}{6}\frac{(3n-7)p-(3n-10)q}{(n-2)p-(n-3)q}.$$

The value of X is now seen to become

$$\frac{1}{\mu}\{\cos^{n-1}\alpha+(n-1)\cos^{n-2}\alpha\sin\alpha(\alpha-\beta)f\}Q=\frac{1}{\mu}\cos^{n-1}\bar{\phi}Q,$$

where

$$\bar{\phi}=\alpha-(\alpha-\beta)f$$

$$=\frac{\alpha+\beta}{2}+(\tfrac{1}{2}-f)(\alpha-\beta)$$

$$=\frac{\alpha+\beta}{2}+\frac{p-q}{(n-2)p-(n-3)q}\cdot\frac{\alpha-\beta}{6}.$$

It is obvious that the value of Y may be obtained in exactly the same way, and that its magnitude is

$$\frac{1}{\mu}\cos^{n-2}\bar{\phi}\sin\bar{\phi}\,Q.$$

§ 7. It is necessary now to enter into an examination of the magnitudes of the errors committed in neglecting squares of the quantity ψ, and to discuss how far (that is to say, for what size of arc) we may with safety employ the values of X and Y just obtained.

Before doing so, I shall briefly remark on the step that we took in neglecting $(p\lambda)^2$ in the value of f. On an examination of equation (9), it will be seen that the principal part of the error committed amounts to

$$-\frac{2n-5}{12}(p\lambda)^2.$$

The error, therefore, in the angle $\bar{\phi}$ is

$$\frac{2n-5}{12}\frac{(p-q)^2}{q^2}(\alpha-\beta).$$

The consequent errors in X and Y, if we put D for the number of degrees in $\alpha-\beta$, are given by

$$\frac{\delta X}{X}=-\tan\bar{\phi}\,.\,\frac{(n-1)(2n-5)}{12}\left(\frac{p-q}{q}\right)^2\frac{\pi}{180}D,$$

$$\frac{\delta Y}{Y}=\text{ a similar expression.}$$

For the sake of simplicity, and to fix our ideas, let us take the case of

$n=3$, and suppose that we are integrating over an arc of 5°, in which one fourth of the horizontal velocity is lost. Then

$$\frac{\delta X}{X} = -\tan\bar{\phi}\cdot\tfrac{1}{3100};$$

$$\frac{\delta Y}{Y} = \cot 2\bar{\phi}\ \tfrac{1}{1550}\ \text{approximately.}$$

It will therefore be observed that the error committed is trifling. As $\bar{\phi}$ will be small over the greater part of any trajectory, this error will chiefly affect Y; but by good fortune the sign of the error is opposite to the error we shall presently find, and therefore helps to neutralize that error.

Returning to the integral for X, we are now to take account of the square of ψ. Let the right-hand side of equation (6) be put equal to

$$\frac{\mu}{g} n \sec^{n+1}\alpha\psi'.$$

Then

$$\psi' = \psi - \frac{n+1}{2}\tan\alpha\psi^2,$$

$$\therefore\ \psi = \psi' + \frac{n+1}{2}\tan\alpha\psi'^2.$$

Substituting this value of ψ in the expansions of $\cos^{n-1}\phi$ and $\cos^{n-2}\phi\sin\varphi$, we get

$$\cos^{n-1}\alpha + (n-1)\cos^{n-2}\alpha\sin\alpha\psi'$$
$$+\frac{n-1}{2}\{2n\sin^2\alpha - 1\}\cos^{n-3}\alpha\psi'^2$$
$$+\text{ etc.,}$$

and

$$\cos^{n-2}\alpha\sin\alpha + \{(n-2)\sin^2\alpha - \cos^2\alpha\}\cos^{n-3}\alpha\psi$$
$$+\{2(n-1)(n-2)\sin^2\alpha - 4(n-1)\cos^2\alpha\}\cos^{n-4}\alpha\sin\alpha\frac{\psi'^2}{2}$$
$$+\text{ etc.}$$

When the expressions just found are substituted in the (C) and (D) integrals, it is obvious that we have to determine these two integrals—

$$R' = \int_q^p \psi'\ \frac{du}{u^{n-1}},$$

$$S' = \int_q^p \psi'^2\ \frac{du}{u^{n-1}}.$$

Referring to the investigation of R in § 6, we see that the work for R′ will be the same, except that instead of $\alpha-\beta$ we should have to put an angle ψ' corresponding to it, such that

$$\psi'=\alpha-\beta-\frac{n+1}{2}\tan\alpha(\alpha-\beta)^2.$$

The error in X due to the difference between R′ and R is ∴

$$-\frac{n^2-1}{2\mu}\cos^{n-3}\alpha\sin^2\alpha(\alpha-\beta)^2\mathrm{Q}f.$$

The integral S′ may be reduced in the same way as the integral R. The most important part of it will be found to be

$$\mathrm{Q}\frac{(\alpha-\beta)^2}{3}.$$

The corresponding error in X is therefore

$$\frac{n-1}{2\mu}(2n\sin^2\alpha-1)\cos^{n-3}\alpha\frac{(\alpha-\beta)^2}{3}\mathrm{Q}.$$

In the former of these two components of the error of X it will be sufficient for the purposes of an estimate to put $f=\frac{1}{2}$. In that case the sum of the two component errors (call it δX) amounts to

$$\frac{n-1}{2\mu}\left\{\frac{2n\sin^2\alpha-1}{3}-\frac{n+1}{2}\sin^2\alpha\right\}\cos^{n-3}\alpha(\alpha-\beta)^2\mathrm{Q}$$

$$=\frac{n-1}{2\mu}\left\{\frac{(n-3)\sin^2\alpha-2}{6}\right\}\cos^{n-3}\alpha(\alpha-\beta)^2\mathrm{Q}.$$

It may be shown in a similar way that the error in Y is given by

$$\delta\mathrm{Y}=\frac{1}{\mu}\left\{\frac{(n-1)(n-3)\sin^2\alpha-5n+11}{12}\right\}\sin\alpha\cos^{n-4}\alpha(\alpha-\beta)^2\mathrm{Q}.$$

A discussion of these expressions for any assigned value of n would determine for what magnitude of arc we might with safety employ the formulæ for X and Y. I shall confine myself to the case when $n=3$, and for the purposes of a ready estimate I shall take

$$\mathrm{X}=\frac{1}{\mu}\cos^2\frac{\alpha+\beta}{2}\mathrm{Q},$$

$$\mathrm{Y}=\frac{1}{\mu}\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha+\beta}{2}\mathrm{Q}.$$

It thus appears that when $n=3$,

$$\frac{\delta X}{X}=-\frac{2}{1+\cos(\alpha+\beta)}\,\frac{(\alpha-\beta)^2}{3},$$

and

$$\frac{\delta Y}{Y}=-\frac{2\tan\alpha}{\sin(\alpha+\beta)}\,\frac{(\alpha-\beta)^2}{3}.$$

Now suppose we were integrating over an arc of 5°, then $\frac{(\alpha-\beta)^2}{3}$ might, approximately, be put equal to $\frac{25}{3\times3281}=\frac{1}{393}$. Our results would therefore be less than $\frac{1}{300}$ in error. Moreover the error in Y, which is really the more important of the two, is less than this, as I have pointed out at the beginning of this article. If, therefore, the formulæ are otherwise serviceable, their inherent errors do not seem to be a great objection to their use.

§ 8. The formulæ for X and Y already found apply only to the ascending branch of the trajectory. A little consideration enables us to see that the same formulæ apply to the descending branch, provided the mean angle $\bar{\phi}$ is

$$\frac{\alpha+\beta}{2}-\frac{p-q}{(n-2)p-(n-3)q}\,\frac{\beta-\alpha}{6},$$

β being now greater than α.

§ 9. In the preceding articles we have neglected all consideration of the time-integral. It may, of course, be treated in the same way as the distance-integrals. I shall not, however, go into the general case, but shall merely state the result in the important case when $n=3$,

$$T=\frac{\cos^2\bar{\phi}'}{\mu}\int_q^p\frac{du}{u^3},$$

where $\bar{\phi}'$ is equal to

$$\frac{\alpha+\beta}{2}+\frac{p-q}{9p-3q}\,\frac{(\alpha-\beta)}{2}$$

in the ascending branch, and

$$\frac{\alpha+\beta}{2}-\frac{p-q}{9p-3q}\,\frac{(\beta-\alpha)}{2}$$

in the descending.

If the arc integrated over is small, $\bar{\phi}'$ is not very different from $\bar{\phi}$.

§ 10. I propose now to collect the results I have proved, stating them in the order and form in which they would be used in conjunction with Mr. Bashforth's tables for K.

Summary of Results when $n=3$, $\mu=\frac{d^2}{\mathrm{W}}\frac{\mathrm{K}}{1000^3}$.

$$\left(\frac{1000}{q}\right)^3-\left(\frac{1000}{p}\right)^3=\frac{d^2}{\mathrm{W}}\frac{\mathrm{K}}{g}(\mathrm{P}_\alpha-\mathrm{P}_\beta). \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(a)}$$

$$\left.\begin{array}{ll} 2\bar{\phi}=\alpha+\beta+\frac{p-q}{p}\frac{\alpha-\beta}{3} \text{ (ascending branch)}, & . \quad . \quad . \\ \text{or} \quad \alpha+\beta-\frac{p-q}{p}\frac{\beta-\alpha}{3} \text{ (descending branch)}, & . \quad . \\ 2\bar{\phi}'=\alpha+\beta+\frac{(p-q)}{9p-3q}(\alpha-\beta) \text{ (ascending branch)}, & . \quad . \\ \text{or} \quad \alpha+\beta-\frac{(p-q)}{9p-3q}(\beta-\alpha) \text{ (descending branch)}; & . \end{array}\right\} \text{(b)}$$

$$\mathrm{X}=\frac{\mathrm{W}}{d^2}(1+\cos 2\bar{\phi})\frac{500{,}000}{\mathrm{K}}\left(\frac{1000}{q}-\frac{1000}{p}\right), \quad . \quad . \quad . \quad . \quad . \qquad \text{(c)}$$

$$\mathrm{Y}=\frac{\mathrm{W}}{d^2}\sin 2\bar{\phi}\,\frac{500{,}000}{\mathrm{K}}\left(\frac{1000}{q}-\frac{1000}{p}\right), \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(d)}$$

$$\mathrm{T}=\frac{\mathrm{W}}{d^2}(1+\cos 2\bar{\phi}')\frac{250}{\mathrm{K}}\left(\overline{\frac{1000}{q}}\Big|^2-\overline{\frac{1000}{p}}\Big|^2\right). \quad . \quad . \quad . \quad . \quad . \qquad \text{(e)}$$

These formulæ might very easily be used if the calculator were furnished with tables of $\frac{1000}{\mathrm{N}}$ and $\left(\frac{1000}{\mathrm{N}}\right)^2$ where N ranges between the magnitudes of the velocity occurring in practice.

Remarks on the Equation giving the Fall of Velocity in an Arc.

§ 11. It will be observed that the two foregoing methods each open with the same equation (a). Now there is a serious difficulty in the use of that equation. Suppose, for example, we were to integrate over an arc of 1°: we should have to use the mean value of K between its values corresponding to the velocities at the beginning and end of the arc. But we do not know the latter of these velocities; it is the very thing we have to find. The first steps in our work must therefore be to guess at it. The practised calculator can, from his experience, make a very good estimate. Having made his estimate, he determines K. He uses this value of K in equation (a); and if he gets the velocity he guessed at, he concludes that he guessed rightly and that he has got the velocity at the end of the arc. If the equation (a) does not agree with him, he makes another guess; and so on, till he comes right. It seems to me, however, that this method of going to work, leaving out of account

the loss of time, is open to objection in the point of accuracy. For, first, there is no method of determining on what principle the mean value of K is to be found—what manner of mean it is. Again, let us suppose for an instant that the velocity at the end of the arc, guessed at, and the value of K are in agreement; that is to say, let the equation

$$\left(\frac{1000}{v_\beta}\right)^3 \sec^3\beta - \left(\frac{1000}{v_\alpha}\right)^3 \sec^3\alpha = \frac{d^2}{W}\frac{K}{g}(P_\alpha - P_\beta)$$

hold for the values of v_β and K used by the calculator. It by no means follows that he has hit on the right v_β and K. For if he is dealing with a part of the tables in which $\frac{dK}{dv}$ happens to be nearly equal to

$$-\frac{3Wg}{d^2}\frac{\sec^3\beta}{P_\alpha - P_\beta}\frac{(1000)^3}{v^4},$$

it is obvious that there are ever so many pairs of values of v_β and K which will stand the test of satisfying the above equation. Now an examination of Mr. Bashforth's tables for ogival-headed shot shows that the value of K diminishes as v increases from 1200 feet upwards, so that $\frac{dK}{dv}$ is negative for a considerable range of values of v which are common in practice. It is not at all unlikely, therefore, that the value for $\frac{dK}{dv}$ just stated may often be very nearly true; in which case the process of guessing becomes extremely dangerous.

I shall in the next method give a plan for determining the velocity at the end of the arc, which seems to me simpler and more satisfactory than the one we have been now discussing. Meanwhile, in connexion with the present method, the following considerations are worthy of notice.

§ 12. Let us consider the fundamental equation

$$\frac{du}{d\phi} = \frac{d^2}{W}\frac{K}{g}\frac{v^4}{\overline{1000}|^3}.$$

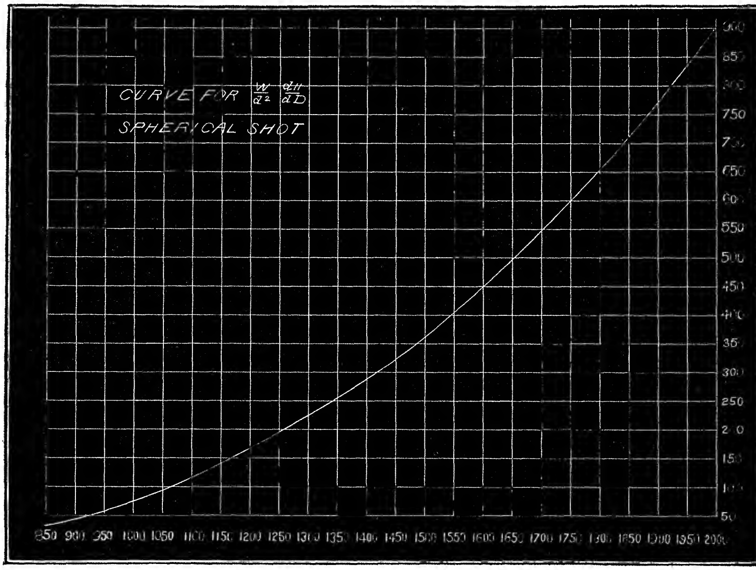
If $\frac{du}{d\phi}$ were uniform, it would be the fall of horizontal velocity in change of inclination equal to the unit of circular measure. As that angle is inconveniently large for discussions connected with trajectories of shot, let us take 1° for unit-angle, and let D be the number of degrees in the angle whose circular measure is ϕ. We then have

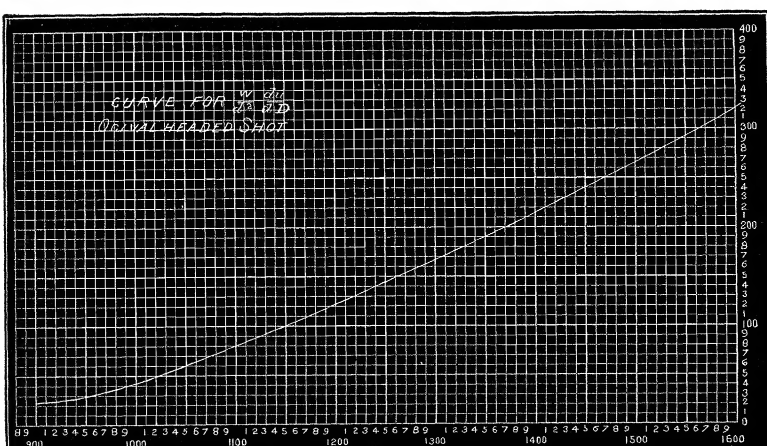
$$\frac{W}{d^2}\frac{du}{dD} = \frac{K}{g}\frac{\pi}{180} \times 1000 \times \left(\frac{v}{1000}\right).$$

The quantity on the right-hand side is the same for all shot of the same kind; and I propose to find its values for intervals of the values of v. It will be convenient to represent those values by the ordinates of a curve

whose abscissæ are the corresponding velocities. Besides the immediate purpose now in view, the same curve will be useful in exhibiting the manner of change of the retardation; since if y be the ordinate corresponding to velocity v, the corresponding retardation will be

$$\frac{d^2}{\mathrm{W}} \frac{180g}{\pi} \frac{y}{v}.$$





The primary use of these curves is that they enable us at a glance to

make a rough estimate of the loss of velocity over an arc of one degree; and we have seen that such an estimate must first be made if we are to employ either of the foregoing methods.

It is possible to make the preliminary guess more near the truth by considering the approximate character of the curves. For example, the curve for ogival-headed shot is approximately a straight line, the tangent of whose inclination from 1700 down to 1250 is ·5; and from 1250 down to 1000, ·4; below 1000 it is unsafe to assign a value. We can easily prove for flat trajectories that the fall of velocity in one degree is either

$$\frac{W}{d^2}\left(\frac{du}{dD}\right)_0\left(\frac{1-e^{-\cdot 5\frac{d^2}{W}}}{\cdot 5}\right) \quad \text{or} \quad \frac{W}{d^2}\left(\frac{du}{dD}\right)_0\left(\frac{1-e^{-\cdot 4\frac{d^2}{W}}}{\cdot 4}\right)$$

between the values above assigned.

Third Method.

§ 13. Returning now to the fundamental equation

$$\frac{du}{d\phi}=\frac{Rv}{g},$$

where R is the retardation due to the resistance of the air, since R is some function of v let us put it equal to $f(v)$. Then, with our old notation still in use, we get by integration

$$g\int_q^p \frac{du}{vf(v)}=\alpha-\beta,$$

$$\therefore\ g\int_q^p \frac{du}{u\sec\phi f(u\sec\phi)}=\alpha-\beta.$$

Now instead of taking some mean value of the quantity K, as was done in the two previous methods, let us make the supposition that the quantity ϕ has its mean value—a supposition in this case by no means extravagant, since for the greater part of the trajectory $\sec\phi$ will vary very slowly. We then have, if D is the number of degrees in $\alpha-\beta$,

$$\frac{180g}{\pi}\int_q^p \frac{du}{u\sec\bar{\phi} f(u\sec\bar{\phi})}=D.$$

Now put $u=V\cos\bar{\phi}$. The equation becomes

$$\frac{180g}{\pi}\int_{q\sec\bar{\phi}}^{p\sec\bar{\phi}} \frac{dV}{Vf(V)}=D\sec\bar{\phi}.$$

With Mr. Bashforth's law substituted, this is

$$\frac{180g}{\pi}(1000)^3\int_{q\sec\bar{\phi}}^{p\sec\bar{\phi}}\frac{dV}{KV^4}=\frac{d^2}{W}D\sec\bar{\phi}. \quad . \quad . \quad . \quad . \quad (a')$$

Now the quantity

$$\frac{180g}{\pi}\overline{1000}|^3\int\frac{dV}{KV^4}$$

can be calculated for every 10 feet, by giving K its mean value over the 10 feet, beginning with 1700 for ogival-headed shot and with 2150 for spherical. Supposing this were done, and the results for all the components of 10 feet added together, the number opposite any velocity v would, for example, in the ogival-head-shot tables, be the value of

$$\frac{180g}{\pi}\overline{1000}|^3\int_v^{1700}\frac{dV}{KV^4}.$$

If we denote this quantity by V_v, the equation (a′) may be written

$$V_{q\sec\bar{\phi}}-V_{p\sec\bar{\phi}}=\frac{d^2}{W}D\sec\bar{\phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (a')$$

which may be regarded as an equation for the determination of q.

The distance-ordinates and time may be found in a similar manner. They are given by

$$\frac{d^2}{W}X=\cos\bar{\phi}\int_{q\sec\bar{\phi}}^{p\sec\bar{\phi}}\frac{\overline{1000}|^3dV}{KV^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b')$$

$$\frac{d^2}{W}Y=\sin\bar{\phi}\int_{q\sec\bar{\phi}}^{p\sec\bar{\phi}}\frac{\overline{1000}|^3dV}{KV^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c')$$

$$\frac{d^2}{W}T=\int_{q\sec\bar{\phi}}^{p\sec\bar{\phi}}\frac{\overline{1000}|^3dV}{KV^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d')$$

If we denote by S_v and T_v the two integrals

$$\int_v^{1700}\frac{\overline{1000}|^3dV}{KV^2} \quad \text{and} \quad \int_v^{1700}\frac{\overline{1000}|^3dV}{KV^3},$$

the equations (b′), (c′), and (d′) become

$$\frac{d^2}{W}X=\cos\bar{\phi}\,(S_{q\sec\bar{\phi}}-S_{p\sec\bar{\phi}}), \quad . \quad . \quad . \quad . \quad . \quad . \quad (b')$$

$$\frac{d^2}{W}\,Y=\sin\bar{\phi}\,(S_{q\sec\bar{\phi}}-S_{p\sec\bar{\phi}}), \quad . \quad . \quad . \quad . \quad . \quad (c')$$

$$\frac{d^2}{W}\,T=T_{q\sec\bar{\phi}}-T_{p\sec\bar{\phi}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (d')$$

The quantities S_v and T_v have been tabulated by Mr. Bashforth for a considerable range of values of v, the upper limit being either 1700 or 2150, according as the shot is ogival-headed or spherical. The tables for the ogival-headed shot have recently been revised and carried to one place further in decimals. The quantities S_v and T_v may therefore be fortunately regarded as completely determined; and the only question will be regarding the mean angle $\bar{\phi}$. Now it is a remarkable circumstance that the value obtained for $\bar{\phi}$ in § 7, if q is not widely different from p, is very nearly the same for all values of n which are not far off from 3. But for limited portions of the trajectory the retardation may be considered as varying according to some simple power of the velocity, though that power is not the same from point to point, but still not far from 3. We may therefore take the value of $\bar{\phi}$ found in § 7 as applicable to the method now in hand. The adoption of this value of the mean angle, since $Y=X\tan\bar{\phi}$, is really equivalent to supposing the shot to move parallel to the chord; and the above proof shows what the limits of integration must be in order that the supposition may be made to approximate to the actual case. The most sensitive quantity in this method, especially near the vertex of the trajectory, is Y; and in finding it over an arc corresponding to a change of inclination as large as 5°, it is necessary to use the correct value obtained in § 7. As regards the other three quantities given by the integrals (a′), (b′), and (d′), it will not matter much if we take $\frac{\alpha+\beta}{2}$, at least for flat trajectories. If, however, the trajectory is not flat, and extreme accuracy is needful, it will be necessary to determine by (a′) the quantity q twice over—first an approximate value of it, in order to get the mean angle $\bar{\phi}$, next its value with the mean angle used in the equation.

The following Table gives the value of V_v for ogival-headed shot as low as 900 feet per second. Below that value of the velocity Mr. Bashforth does not give tables for the resistance, and the magnitudes of the resistance for the lower velocities of ogival-headed shot have yet to be found.

Table of Values of $V_v = \frac{180g}{\pi}(1000)^3 \int_v^{1700} \frac{dV}{KV^4}$.

	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
90...	8·3380	8·2955	8·2529	8·2104	8·1678	8·1253	8·0827	8·0402	7·9976	7·9551
91...	7·9125	7·8714	7·8302	7·7890	7·7479	7·7066	7·6654	7·6242	7·5830	7·5418
92...	7·5006	7·4627	7·4248	7·3868	7·3489	7·3110	7·2730	7·2351	7·1972	7·1592
93...	7·1213	7·0851	7·0489	7·0127	6·9766	6·9404	6·9043	6·8681	6·8319	6·7958
94...	6·7596	6·7251	6·6906	6·6561	6·6216	6·5870	6·5525	6·5180	6·4835	6·4490
95...	6·4146	6·3819	6·3492	6·3166	6·2839	6·2513	6·2186	6·1860	6·1533	6·1207
96...	6·0880	6·0571	6·0263	5·9954	5·9646	5·9338	5·9030	5·8722	5·8413	5·8105
97...	5·7797	5·7507	5·7217	5·6927	5·6637	5·6347	5·6057	5·5767	5·5477	5·5187
98...	5·4898	5·4625	5·4352	5·4079	5·3806	5·3533	5·3260	5·2987	5·2715	5·2442
99...	5·2169	5·1914	5·1660	5·1405	5·1151	5·0896	5·0641	5·0387	5·0132	4·9878
100...	4·9623	4·9386	4·9149	4·8912	4·8675	4·8437	4·8200	4·7963	4·7726	4·7489
101...	4·7252	4·7032	4·6812	4·6591	4·6371	4·6151	4·5931	4·5711	4·5490	4·5270
102...	4·5050	4·4847	4·4643	4·4440	4·4236	4·4033	4·3830	4·3626	4·3423	4·3219
103...	4·3016	4·2829	4·2642	4·2456	4·2269	4·2082	4·1895	4·1708	4·1522	4·1335
104...	4·1148	4·0977	4·0806	4·0635	4·0464	4·0293	4·0122	3·9951	3·9780	3·9609
105...	3·9439	3·9282	3·9125	3·8968	3·8811	3·8655	3·8498	3·8341	3·8184	3·8027
106...	3·7871	3·7726	3·7581	3·7436	3·7292	3·7147	3·7002	3·6857	3·6712	3·6567
107...	3·6422	3·6287	3·6151	3·6016	3·5881	3·5746	3·5610	3·5475	3·5340	3·5204
108...	3·5069	3·4941	3·4814	3·4686	3·4558	3·4431	3·4303	3·4175	3·4047	3·3920
109...	3·3792	3·3670	3·3549	3·3427	3·3306	3·3184	3·3062	3·2941	3·2819	3·2698
110...	3·2576	3·2460	3·2344	3·2228	3·2111	3·1995	3·1879	3·1762	3·1646	3·1529
111...	3·1413	3·1301	3·1190	3·1078	3·0966	3·0855	3·0743	3·0631	3·0519	3·0408
112...	3·0296	3·0189	3·0081	2·9974	2·9867	2·9760	2·9652	2·9545	2·9438	2·9330
113...	2·9223	2·9120	2·9017	2·8913	2·8810	2·8707	2·8604	2·8501	2·8397	2·8294
114...	2·8191	2·8092	2·7992	2·7893	2·7794	2·7695	2·7595	2·7496	2·7397	2·7297
115...	2·7198	2·7102	2·7007	2·6916	2·6816	2·6721	2·6625	2·6530	2·6435	2·6339
116...	2·6244	2·6152	2·6059	2·5967	2·5874	2·5782	2·5689	2·5597	2·5504	2·5412
117...	2·5319	2·5230	2·5141	2·5052	2·4963	2·4874	2·4785	2·4696	2·4607	2·4518
118...	2·4429	2·4343	2·4257	2·4171	2·4085	2·3999	2·3913	2·3827	2·3741	2·3655
119...	2·3570	2·3486	2·3403	2·3320	2·3237	2·3154	2·3071	2·2988	2·2905	2·2822
120...	2·2739	2·2659	2·2578	2·2498	2·2418	2·2338	2·2257	2·2177	2·2097	2·2016
121...	2·1936	2·1858	2·1781	2·1703	2·1625	2·1548	2·1470	2·1392	2·1314	2·1237
122...	2·1159	2·1084	2·1008	2·0933	2·0858	2·0783	2·0707	2·0632	2·0557	2·0481
123...	2·0406	2·0333	2·0260	2·0188	2·0115	2·0042	1·9969	1·9896	1·9824	1·9751
124...	1·9678	1·9607	1·9537	1·9466	1·9396	1·9325	1·9254	1·9184	1·9113	1·9043
125...	1·8972	1·8903	1·8834	1·8766	1·8697	1·8629	1·8561	1·8492	1·8424	1·8355
126...	1·8287	1·8221	1·8154	1·8088	1·8022	1·7956	1·7889	1·7823	1·7757	1·7690
127...	1·7625	1·7560	1·7496	1·7432	1·7367	1·7303	1·7239	1·7174	1·7110	1·7045
128...	1·6981	1·6919	1·6856	1·6794	1·6731	1·6669	1·6606	1·6544	1·6481	1·6419
129...	1·6356	1·6295	1·6234	1·6174	1·6113	1·6052	1·5991	1·5931	1·5870	1·5810
130...	1·5749	1·5689	1·5630	1·5571	1·5512	1·5453	1·5394	1·5334	1·5275	1·5216
131...	1·5157	1·5099	1·5042	1·4984	1·4927	1·4870	1·4813	1·4756	1·4698	1·4641
132...	1·4584	1·4528	1·4472	1·4417	1·4361	1·4305	1·4249	1·4193	1·4138	1·4082
133...	1·4026	1·3972	1·3918	1·3863	1·3809	1·3755	1·3701	1·3646	1·3592	1·3538
134...	1·3484	1·3431	1·3378	1·3325	1·3272	1·3220	1·3167	1·3114	1·3061	1·3008
135...	1·2955	1·2904	1·2852	1·2801	1·2749	1·2698	1·2646	1·2595	1·2543	1·2492
136...	1·2440	1·2390	1·2340	1·2289	1·2239	1·2189	1·2139	1·2089	1·2038	1·1988
137...	1·1938	1·1889	1·1840	1·1791	1·1742	1·1693	1·1644	1·1595	1·1546	1·1497
138...	1·1448	1·1400	1·1352	1·1305	1·1257	1·1209	1·1161	1·1113	1·1066	1·1018
139...	1·0970	1·0923	1·0876	1·0829	1·0783	1·0736	1·0689	1·0643	1·0596	1·0549
140...	1·0503	1·0457	1·0412	1·0366	1·0321	1·0275	1·0229	1·0184	1·0138	1·0093
141...	1·0047	1·0002	·9958	·9913	·9869	·9824	·9779	·9735	·9690	·9646
142...	·9601	·9557	·9514	·9470	·9427	·9383	·9339	·9296	·9252	·9209
143...	·9165	·9122	·9079	·9036	·8994	·8951	·8908	·8866	·8823	·8780
144...	·8737	·8695	·8653	·8612	·8570	·8528	·8486	·8444	·8402	·8361
145...	·8319	·8278	·8237	·8196	·8155	·8114	·8073	·8032	·7991	·7950

Table (*continued*).

	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
146...	·7910	·7870	·7830	·7790	·7750	·7710	·7669	·7629	·7589	·7549
147...	·7509	·7470	·7430	·7391	·7352	·7312	·7273	·7234	·7194	·7155
148...	·7115	·7077	·7038	·6999	·6961	·6922	·6884	·6845	·6806	·6768
149...	·6729	·6691	·6654	·6616	·6578	·6540	·6502	·6465	·6427	·6389
150...	·6351	·6314	·6277	·6239	·6202	·6165	·6128	·6091	·6053	·6016
151...	·5979	·5943	·5906	·5870	·5833	·5797	·5760	·5724	·5687	·5651
152...	·5614	·5578	·5542	·5506	·5470	·5435	·5399	·5363	·5327	·5291
153...	·5255	·5220	·5185	·5149	·5114	·5079	·5044	·5009	·4973	·4938
154...	·4903	·4868	·4834	·4799	·4765	·4730	·4695	·4661	·4626	·4592
155...	·4557	·4523	·4489	·4455	·4421	·4387	·4352	·4318	·4284	·4250
156...	·4216	·4183	·4149	·4116	·4082	·4049	·4015	·3982	·3948	·3915
157...	·3881	·3848	·4815	·3782	·3749	·3716	·3684	·3651	·3618	·3585
158...	·3552	·3520	·3487	·3455	·3422	·3390	·3357	·3325	·3292	·3260
159...	·3227	·3195	·3163	·3132	·3100	·3068	·3036	·3004	·2973	·2941
160...	·2909	·2877	·2846	·2814	·2783	·2751	·2719	·2688	·2656	·2625
161...	·2593	·2562	·2531	·2500	·2469	·2439	·2408	·2377	·2346	·2315
162...	·2284	·2254	·2223	·2193	·2162	·2132	·2102	·2071	·2041	·2010
163...	·1980	·1950	·1920	·1890	·1860	·1831	·1801	·1771	·1741	·1711
164...	·1681	·1652	·1622	·1593	·1563	·1534	·1505	·1475	·1446	·1416
165...	·1387	·1358	·1329	·1300	·1271	·1243	·1214	·1185	·1156	·1127
166...	·1098	·1070	·1042	·1013	·0985	·0957	·0929	·0900	·0872	·0844
167...	·0816	·0788	·0760	·0732	·0705	·0677	·0649	·0621	·0594	·0566
168...	·0538	·0511	·0484	·0456	·0429	·0402	·0375	·0347	·0320	·0293
169...	·0266	·0239	·0213	·0186	·0159	·0133	·0106	·0080	·0053	·0027

An example of the use of these Tables was given in the abstract which was printed in the 'Proceedings' of the Society, vol. xxv. p. 18.

Postscript.

Professor J. Couch Adams, to whom this paper was shown before publication, has obtained a solution of the equations employed in the second method, which is of a more complete and satisfactory character than the one given above. The results he has arrived at are contained in the following note, which he has kindly allowed me to subjoin:—

"Employing the notation of the paper, and supposing the resistance to vary as the nth power of the velocity, the horizontal velocity q is given by the equation

$$\frac{1}{q^n}-\frac{1}{p^n}=\frac{\mu n}{g}\left(\sec\frac{\alpha+\beta}{2}\right)^{n+1}(\alpha-\beta)\left\{1+\tfrac{1}{24}(n+1)\left[(n+2)\left(\sec\frac{\alpha+\beta}{2}\right)^2-(n+1)\right](\alpha-\beta)^2\right.$$

where $\alpha-\beta$ is expressed in the circular measure.

"The inclination $\bar{\phi}$ of the chord AB is given by

$$\bar{\phi}=\frac{\alpha+\beta}{2}+\tfrac{1}{3}\frac{p-q}{p+q}(\alpha-\beta)+\tfrac{1}{4}\left(\tan\frac{\alpha+\beta}{2}\right)(\alpha-\beta)^2,$$

where $\alpha-\beta$ is supposed to be expressed as before in the circular measure. If $\alpha-\beta$ be expressed in minutes, the last term must be multiplied by $\sin 1'$.

"The value of the mean angle $\overline{\phi'}$ to be employed in finding the time-integral is

$$\overline{\phi'}=\frac{\alpha+\beta}{2}+\frac{1}{6}\frac{p-q}{p+q}(\alpha-\beta)+\frac{1}{4}\left(\tan\frac{\alpha+\beta}{2}\right)(\alpha-\beta)^2,$$

where the last term is the same as that in the above value of $\overline{\phi}$, but the second term is only one half of its amount in the former case.

"It will be seen that the above expressions for $\overline{\phi}$ and $\overline{\phi'}$ are independent of the value of n.

"Also, if

$$Q=1-\frac{1}{24}(n-1)\left[(n-2)\left(\sec\frac{\alpha+\beta}{2}\right)-(n-3)\right](\alpha-\beta)^2,$$

where $\alpha-\beta$ is expressed as before in the circular measure, the values of the coordinates X, Y, and of the time T are given by

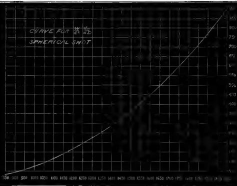
$$X=\frac{1}{\mu(n-2)}Q(\cos\overline{\phi})^{n-1}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right),$$

$$Y=\frac{1}{\mu(n-2)}Q(\cos\overline{\phi})^{n-2}\sin\overline{\phi}\left(\frac{1}{q^{n-2}}-\frac{1}{p^{n-2}}\right)=X(\tan\overline{\phi}),$$

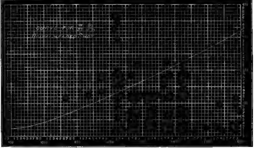
$$T=\frac{1}{\mu(n-1)}Q(\cos\overline{\phi'})^{n-1}\left(\frac{1}{q^{n-1}}-\frac{1}{p^{n-1}}\right).$$

"It may be remarked that if α and β, as well as p and q, be interchanged, the values of $\overline{\phi}$, $\overline{\phi'}$, and Q will remain unaltered, and the values of X, Y, and T will merely change their signs, as it is evident should be the case.

"The above values of $\overline{\phi}$, $\overline{\phi'}$, and Q are true to the third order of small quantities inclusive, and the values of X, Y, and T are true to the fourth order, considering $\frac{p-q}{p+q}$ and $\alpha-\beta$ to be small quantities of the first order."



CURVE FOR
SPHERICAL SHOT


CURVE FOR